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Correlation Dimension for Paired Discrete Time Series

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We present a method for detecting the dimension of a dynamical system encompassing simultaneously two distinct discrete time series. This method is an extension of the technique introduced by Grassberger and Procaccia for single time series and allows to evaluate the common correlation dimension of the chaotic attractor. The method is applied to some mathematical models and to multiple single-neuron spike trains.

KEY WORDS: Chaotic attractors; correlation dimension; spike trains.

1. INTRODUCTION

The problem of the detection and characterization of chaotic attractors in discrete time series has been widely investigated in recent years. Dynamical system theory provides constructive algorithms to discern deterministic or stochastic behaviour,[8, 5] and allows to measure the size of the attractor.[4, 12, 17] Discrete time series may be derived from mappings, differential equations or experimental data. We will consider the conservative 2D-standard map and a 4D-mapping derived from the coupling of dissipative 2D-Hénon mappings. Moreover we will examine a neurobiological application, where the time series is formed by the occurrences of spikes during electrophysiological recordings ("spike trains").

Nowadays the most reliable method to determine the random or deterministic character of a time series is provided by ref. [8] (hereafter

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GP-algorithm). This method has been widely applied to mathematical models[8] as well as to biological data.[2, 11, 13, 3] In ref. [3] we analyzed a set of spike–train samples using the GP–algorithm and found several low–dimensional chaotic attractors. In this paper we extend these results, considering the problem of finding a deterministic structure associated simultaneously to two time series (see also ref. [10]), derived from a single observable (taking for example distinct initial conditions) or from different observables measured simultaneously. To this end, we develop an extension of the GP-algorithm and we apply it to the standard map, the 4-D Hénon mapping and to neuronal data.

The "double series" method provides good results when two small datasets are available, instead of a single (long) time series. This situation often applies when dealing with experimental data.

2. SINGLE AND MULTIPLE TIME SERIES ANALYSIS

We briefly recall the GP-algorithm as follows. Let $x_1, ..., x_K$ (where K is the total number of points) be a given time series; we define *delay coordinates* $\{y_1, ..., y_N\}$ $(N = K - d + 1)$ in a d-dimensional embedding space, setting $y_j = (x_j, ..., x_{j+d-1})$, $j = 1, ..., N$. For a given $r > 0$, we define the *correlation integral* as

$$C_{N,d}(r) \equiv \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} \Theta(r - |y_i - y_j|_d), \tag{C}$$

where Θ is the Heaviside function and $|\cdot|_d$ denotes the Euclidean norm in $\mathbf{R}^d$. The *correlation dimension* v is related to the above function by

$$v \equiv \lim_{r \to 0} \lim_{N \to \infty} \frac{d \log C_{N,d}(r)}{d \log r} \tag{1}$$

for d sufficiently large. Therefore the correlation dimension corresponds to the slope of the curves $\log C_{N,d}(r)$ vs. $\log r$, whenever its value becomes nearly constant as the embedding dimension is varied. In practical applications, the slope of the above curves must be evaluated in a meaningful range of values of the radius, say (r_0, r_1), referred to as the *scaling region*: below r_0 the curves are distorted due to poor statistics, while above r_1 the curves tend to flatten since the attractor has finite size. Particular care must be devoted to the selection of the scaling region, whose amplitude should be sufficiently large.[6]

We assume that two series, say $\{x_j^{(1)}\}$ and $\{x_j^{(2)}\}$ with K_1 and K_2 points respectively, are distinct and independent realizations of the same observable in a given system or that they represent different observables measured simultaneously. The GP-algorithm can be extended replacing the definition of the correlation integral (C) as follows. Let $\{y_j^{(1)}\}$, $\{y_j^{(2)}\}$ be the delay coordinates associated to $\{x_j^{(1)}\}$, $\{x_j^{(2)}\}$ in a d-dimensional embedding space. Let $\{\tilde{y}_i\}$ be the sequence defined by $\tilde{y}_i = y_i^{(1)}$ $(1 \leqslant i \leqslant N_1)$ and $\tilde{y}_i = y_{i-N_1}^{(2)}$ $(N_1 + 1 \leqslant i \leqslant N_1 + N_2)$, where $N_1 = K_1 - d + 1$, $N_2 = K_2 - d + 2$. We define the *correlation integral* associated to $\{x_j^{(1)}\}$, $\{x_j^{(2)}\}$ as

$$C_{N_1+N_2,d}(r) = \frac{1}{(N_1+N_2)^2} \sum_{k=1}^{2} \sum_{j=1}^{N_k} \sum_{i \neq j, i=1}^{N_1+N_2} \Theta(r - |y_j^{(k)} - \tilde{y}_i|). \quad (C_{12})$$

Therefore for any point of each series, one counts the numbers of points (from *both* series) falling inside a hypersphere of radius r around the center. The above formula can be easily extended to an arbitrary number of time series. We remark that (C_{12}) reduces to (C) as $N_1 = 0$ or $N_2 = 0$. More precisely, the above definition is equivalent to split a discrete series of length N into two series of lengths N_1 and N_2. However when the attractor (if it does exist) is *curly* shaped (e.g., successive components of the discrete series are far apart), the double trajectory method might become more efficient than the original GP-algorithm, whenever the two series are close to each other.

In order to test the validity of this method we consider two mathematical models. The first one is the standard mapping, described by the set of equations

$$\begin{cases} y_{j+1} = y_j + \varepsilon \sin x_j & x_j \in \mathbf{R}/2\pi\mathbf{Z} \\ x_{j+1} = x_j + y_{j+1} & y_j \in \mathbf{R}, \quad \varepsilon \in \mathbf{R}_+. \end{cases} \quad (SM2)$$

The global breakdown threshold for transition to chaos was estimated in ref. [9] as $\varepsilon \simeq 0.9716$. Therefore, in order to select (strongly) chaotic trajectories we fix $\varepsilon = 1.2$ and we choose initial conditions which do not correspond to librational curves nor to periodic orbits. We define the time series as the iterates $\{x_j\}$ of $(SM2)$. The application of the single and multiple time series-methods shows that the embedding and correlation dimensions can be computed more efficiently using the double series method. In fact, with a single time series of 6000 points we find that the slopes of the correlation integrals are convergent starting from $d = 3$ (see Fig. 1*a*), while we should clearly find $d = 2$. Much longer time series should be considered to get the right result. On the contrary, taking two time series, with $K_1 = K_2 = 1000$, we are able to confirm that both trajectories lie in a 2–dimensional embedding space with a correlation dimension about equal to $\nu = 2$ (Fig. 1*b*). This last value is consistent with the theoretical value one expects from the ergodic behaviour of chaotic trajectories.

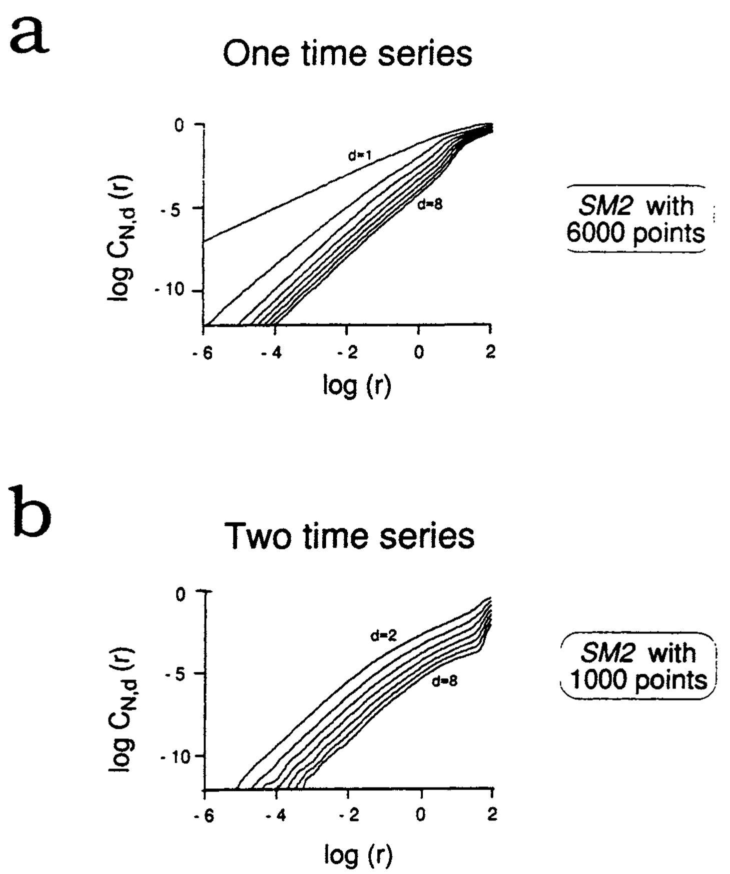

Fig. 1. (a) Two-dimensional standard-map ($SM2$) with $\varepsilon = 1.2$, initial conditions: ($x = 0$, $y = 0.61877$). The correlation integral is plotted for $d = 1, \ldots, 8$; $K = 6000$. (b) Double time series of $SM2$ with $\varepsilon = 1.2$, initial conditions: ($x_0 = 0$, $y_0 = 0.61877$) and ($x_1 = 0$, $y_1 = 0.62355$); $K = 1000$.

We test our method also on a 4D-mapping, derived from the coupling of two 2D-Hénon's maps. More precisely, we construct a new mapping defined by

$$\begin{cases} x_{j+1} = -ax_j^2 + y_j + 1 \\ y_{j+1} = bx_j + hz_j \\ z_{j+1} = -a'z_j^2 + t_j + 1 \\ t_{j+1} = -b'z_j + hx_j \end{cases} \qquad (HE4)$$

where x_j, y_j, z_j, $t_j \in \mathbf{R}$, a, b, a', b', h are real parameters (that we fix equal to $a = 1.4$, $a' = 1.6$, $b = 0.3$, $b' = 0.6$, $h = 0.03$). Notice that for $h = 0$, $(HE4)$ reduces to two uncoupled Hénon's mappings. The single time series method provides an embedding space of dimension $d = 3$, taking 1000, 2000 or 4000 points. On the contrary, with an overall number of 2000 points (computed on iterations of coordinates x and z), one obtains that the two trajectories are embedded in a 4-dimensional space and there exists an attractor of dimension $\nu \simeq 2.03$. The effect of the (weak) coupling is observed using both x and z in the double series method, rather than considering iterations of a single coordinate. We found similar results in $(SM2)$ and $(HE4)$ for different values of the parameters.

As a further application, we compute the correlation dimension for neuronal pairs, where time series correspond to the "spike train" sequences formed by the occurrences of spikes during simultaneous extracellular recording of the activity of two single units. More specifically we want to show that the experimental time series-spike trains-correspond to different observables measured simultaneously in the same system.

The data are derived from experiments conducted in 5 anesthetized young adult Long–Evans rats in the substantia nigra pars reticulata (13 spike trains) or in the auditory thalamus (14 spike trains).[15, 16] The recordings were performed during spontaneous activity and acoustically evoked stimulation with an accuracy of 1 *ms*.[14]

Low-dimensional deterministic dynamics with an embedding dimension between 2 and 6 and a correlation dimension between 0.14 and 3.3, was observed in 7 over 27 single spike trains, containing between 800 and 5200 points.[3] In this paper we implement the double series method, taking all possible combinations ($n = 68$) of pairs within the same groups of cells provided by the 27 spike trains analyzed in ref. [3]. We have found evidence of deterministic behaviour for 10 pairs with an embedding dimension ranging from 3 to 6 and a correlation dimension between 0.27 and 3.7. In most cases (7/10) the deterministic coupled dynamics was observed when one point process of the pair was also characterized by a deterministic structure. In two cases the generating processes were not deterministic and only one significant coupled dynamics was characterized by a pair of generating processes.

It is interesting to note that only one significant case was observed in substantia nigra pars reticulata. The majority of significant dynamics for neuronal pairs was observed in a cell group recorded in the auditory thalamus: 5 cases were found under external stimulus and 4 cases under spontaneous activity. As an example, we show in Fig. 2 the analysis of a triplet of cells during binaural white noise stimulation. Firstly, the response pattern to the stimulus, depicted by the peri-stimulus time histogram, is

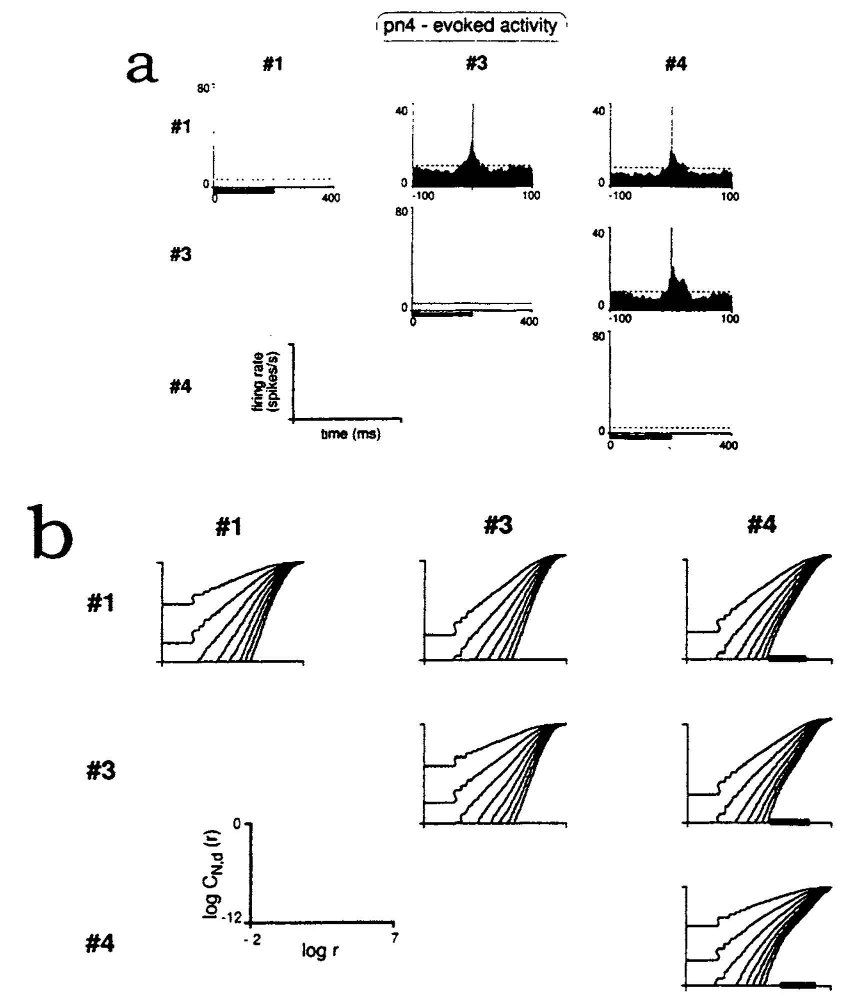

Fig. 2. Single unit activity of a triplet of cells in the auditory thalamus during evoked activity. Row and column numbers indicate the cell label. The number of points in the time series and the average firing rate (spikes/s) for the cells are the following: cell 1 (3270, 8.2); cell 2 (3178, 7.9); cell 3 (2655, 6.6). (a) Time domain analysis. The panels on the diagonal correspond to the peri-stimulus time histograms with time (ms) vs. instantaneous firing rate (spikes/s) scaled as in ref. [1]. The remaining panels refer to the crosscorrelograms. The ordinate corresponds to the instantaneous firing rate of the follower cell (column number) at variable delays before (negative times on the abscissa, up to −100 ms) and after (positive times, up to 100 ms) the firing of the trigger cell (row number). Curves were smoothed using a Gaussian convolution with a bin size equal to 2.5 ms. Broken lines indicate the upper 99% confidence limit. (b) Dynamical system analysis. In case of significant deterministic dynamics, the scaling region is marked by a thick line on the abscissa. The curves which refer to one cell (single time series) form the panels on the diagonal. The remaining curves refer to two spike trains recorded simultaneously.

different for each cell (Fig. 2*a*). All cells were characterized by a transient onset excitatory response. In addition, at the offset of the stimulus the activity of cell 3 was inhibited, whereas cell 4 exhibited an excitatory response (Fig. 2*a*). Cross-correlograms[7] between all pairs (upper triangular part of Fig. 2*a*) show a peak near time zero, thus suggesting that the spike trains are synchronous. On the opposite, only the dynamics of pairs (1; 4) and (3; 4) was deterministic (Fig. 2*b*). This suggests that coupling of pair (1; 3) is independent from the activity of cell #4.

3. DISCUSSION

An alternative method for the detection of a deterministic structure in discrete time series has been proposed. The technique is based on the analysis of the dynamical properties associated simultaneously to two time series, obtained either taking distinct initial conditions of the same observable or different observables measured at the same time in the same system. We have investigated the existence of dynamics associated to two time series in mathematical models and biological applications.

The advantage of using the double series method has been confirmed by the applications on the 2D-conservative standard map ($SM2$) and on the dissipative Hénon's like 4D-mapping ($HE4$). The application of this method to time series provided by spike trains is interesting for several reasons. The detection of a deterministic dynamics necessarily requires the stability of the generating processes over a relatively long period of time and the double series method may provide useful results over a limited recording time. In addition, the application of dynamical systems methods to spike train interactions could be extended to a larger number of simultaneoulsy recorded spike trains and offers as a complementary approach toward deciphering the neural code.

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